An Optimal Modelling for Unsaturated Seepage

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Abstract For groundwater and moisture seepage, optimization of locations or shapes of hydraulic constructions (cavities, tunnels, etc.) differs from the classical fluid dynamics. In this paper, a ponded seepage problem is considered to get an optimal depth of drains. An optimal shape design is considered using a new (mathematical) approach.

1. INTRODUCTION

With the advent of FDM - FEM packages like MODFLOW, analytic solutions for groundwater flow problems became a supplementary tool in engineering practice. However, new environments like *Mathematica* allow reconsideration of the applicability of 'old-fashioned' analytic techniques which restore from seemingly ponderous forms into standard built-in computer operations. In this paper, we study mainly optimization problems which arise in seepage into drains and cavities. Our goal is to derive some new solutions. We focus our interest on one of the most important characteristics, total rate of water seeping into a drain, even though other distributed (flow nets, specific discharge, moisture, etc.) or integral (erosion safety factors) characteristics can be analyzed in a similar way. For some specific flow patterns we answer the following questions: Is there an optimal tunnel depth providing minimal rate for ponded conditions? What is the influence of cavity shape on the rate and is there an optimal form providing minimal rate under imposed isoperimetric restrictions? In section 3, we will show the new technique for deriving the necessary condition for optimality; this part is, in turn, the sensitivity analysis, too. We will show the full process of derivation, since that will help readers to be familiar with the method more easily. The main tools used are Taylor expansions, integration by parts, and Green's formula which civil engineeres know well. Throughout the paper, we assume steady Darcian flows of incompressible one-phase fluids in rigid porous media.

2. PONDED SEEPAGE

In this section we treat problems of ponded seepage in order to illustrate usefulness of *Mathematica*. Consider an empty drain of radius r located in a homogeneous, isotropic half-plane at the depth c under the soil surface ponded by water with depth H. In the flow domain the hydraulic head $h(x,y)$ satisfies the Laplace equation. We are interested in the value of total seepage rate q_e into the drain and its difference from the rate q_f of a drain filled with water (see Fig. 1).

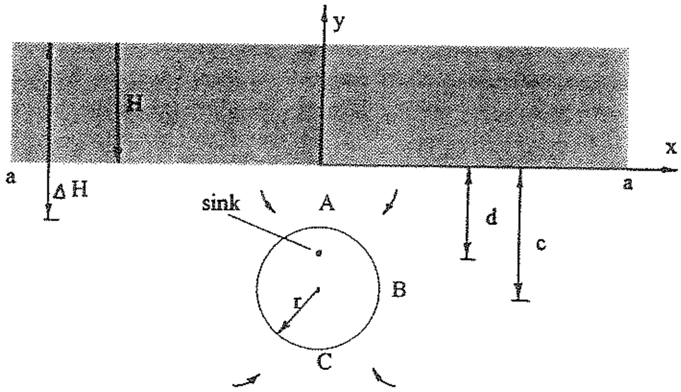

Fig. 1

For the case of a filled drain, Forchheimer found a formula in 1889 (Forchheimer, 1930) for q_f. Polubarinova-Kochina (1977, p.354) approximates the formula as $q_f = 2\pi\Delta H/\ln(2c/r)$, where ΔH denotes the head difference between the soil surface and the drain contour.

For the case of empty drain (tunnel) Freeze and Cherry (1979) say that the only theoretical formula by Goodman et al. for the rate is

$$q_e = 2\pi\Delta H/[2.3\ln(2c/r)].$$

For the steady regime this formula seems to be suspicious because it leads to the contradiction $q_e < q_f$.

To derive the correct formula for an empty drain we utilize the 'sink-solution' (Polubarinova - Kochina, 1977):

$$\phi(x,y) = -h = \frac{q}{4\pi}\ln\frac{x^2+(y-d)^2}{x^2+(y+d)^2} \qquad (1)$$

where ϕ is the velocity potential, q and d are the sink strength and depth respectively. Set $\phi = 0$ along the soil surface. Then along the contour of an empty drain $p = \phi + y - H$. In contrast with the case of a filled drain we search not for an equipotential but for an isobar $p = 0$ as the drain contour. According to (1), we can find the contour equation for the isobar $p = 0$ and can find the drain area S.

Thanks to *Mathematica*, we can plot the curve of $f_1 = 2q_e/\sqrt{S}$. For example, for $q/H = 100$ we found a minimum $f_1 = 5.97$ at $c/H = 15.75$. From the graphs for $10q_f/q_e$ as a function of c/H, we can see (Fig. 2) that the maximal influence of water level in the drain (minimal value of q_f/q_e is 0.29) occurs at $c/H = 9.29$. In other words, at this depth influence of the boundary condition along the drain contour on the rate is most pronounced. Obviously, if the drain is partially filled its rate q_p can be estimated as $q_f < g_p < q_e$ (Ilyinsky and Kacimov, 1992a).

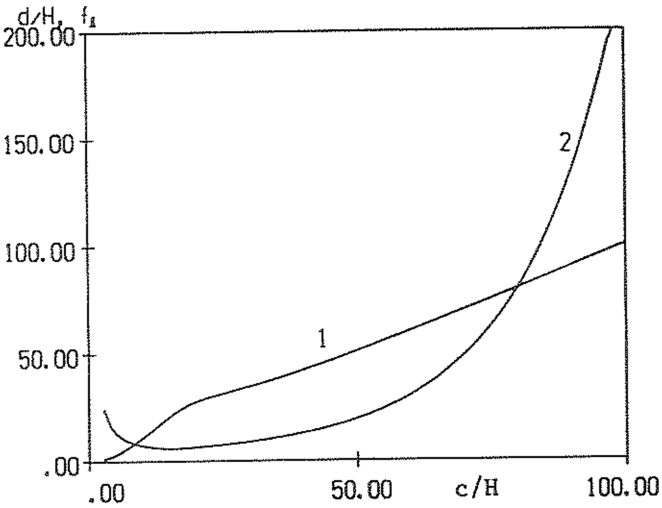

Fig. 2
Curve 1 shows d/H. Curve 2 shows f_1

3. SHAPE OPTIMIZATION

3.1 Unsaturated Flow

In this section, we consider unsaturated flow in terms of the quasi-linear model and dwell on the effect of the drain shape on the rate value starting with the classical Wilson singularity. A comprehensive review of the model we use was made by Pullan (1990), Clothier et al (1995). According to the model, the conductivity k varies exponentially with pressure head h, i.e., $k = k_0 e^{\alpha h}$ where the constants k_0 and α are saturated conductivity and sorptivity, respectively. The governing equation for the matrix flux potential $\phi = K/\alpha$ is

$$\Delta\phi - \alpha\frac{\partial\phi}{\partial z} = 0 \qquad (2)$$

where z is the vertical coordinate oriented downward.

Consider a single drain placed 'near' the origin $(0, 0)$ of the coordinate system in an infinite porous medium. Sufficiently far from the drain, $\phi = k_\infty/\alpha$; i.e., the porous matrix is at constant pressure with head $h = \alpha^{-1}\ln(k_\infty/k_0)$ (obviously, $k_\infty < k_0$ to guarantee purely unsaturated seepage). The vertical and horizontal components of velocity are given by

$$v_z = \alpha\phi - \frac{\partial\phi}{\partial z}, \quad v_x = -\frac{\partial\phi}{\partial x}. \qquad (3)$$

At infinity we have a uniform descendant flow with velocity k_∞.

Assume that along the drain contour Γ pressure is constant and negative. Hence the corresponding potential ϕ_c is k_c/α (obviously, $k_c < k_\infty$ should be valid to provide drainage).

Since the classical studies of ancient Greeks, Saint-Venant, and Rayleygh shape optimization and corresponding isoperimetric inequalities have been investigated in many applications (Pólya and Szegö 1951, Pironneau 1984, Fujii 1990), in particular, for seepage flows(Philip et al. 1989, Ilynsky and Kacimov 1992b). The general statement is clear: what form of the boundary (or its part) of the flow domain provides extreme (minimal or maximal) flow characteristic (say, rate, uplift force, wetted area, etc.) at prescribed isoperimetric restrictions (for example, length, area or volume). In the section devoted to 3-D saturated flows we have mentioned one of the results that dates back to Poincaré (a sphere in an unrestricted aquifer provides minimal rate in the class of equipotential bodies of prescribed volume). In what follows we treat one of the problems of this kind for 2-D unsaturated flow in terms of the quasi-linear model. Unlike the cases above we consider an irrigation cavity which wets the surrounding porous matrix. To our knowledge, in the analytic solutions (Concer 1959, Philip 1984) for a single cavity wetting unrestricted soil only circular and elliptical contours were investigated.

3.2 Problem Statement

Let us study seepage from the contour Γ of constant moisture (pressure) with potential $\phi = \phi_c$ to soil. At infinity $\phi = \phi_\infty, \phi_\infty < \phi_c$. Γ confines the domain Ω outside the cavity and, at the same time it confines $\Gamma^c \equiv R^2 - \Gamma$ of area S. Outside the cavity (namely in Ω) the potential satisfies the steady state infiltration equation (2) with moisture velocities (3). Designate the total seepage rate as q_u.

We want to determine the shape which provides minimal q_u (a criterion) at prescribed S (an isoperimetric restriction), ϕ_c, ϕ_∞. First, consider the case of arbitrary cavities (though with sufficiently smooth contours).

It is well known that minimum should satisfy a necessary condition of optimality (for a function of one variable $C(c)$ this states $dC/dc = 0$ where C is the criterion and c is the control variable) and a sufficient one ($d^2C/dc^2 > 0$). In our case cavity shape is the control function and the criterion depends on infinite number of variables. It calls for subtle methods of boundary variations one of which is presented below.

In what follows we derive a necessary condition for minimum.

$$J(\Omega) = q_u = \oint_\Gamma v_n(s)\,ds$$
$$= -\oint_\Gamma \nabla\phi\cdot\vec{n}\,ds + \alpha\oint_\Gamma \phi n_z\,ds, \qquad (4)$$

where v_n designates the normal component of velocity, i.e., $v_n = \vec{v}\cdot\vec{n}$, $\vec{n} = (n_z, n_x)$ is the inward normal (from the cavity to soil), and s is the arc length of the contour (counterclockwise). In (4), of course, ϕ is the solution of the following boundary value problem:

$$\Delta\phi - \alpha\frac{\partial\phi}{\partial z} = 0 \qquad (\text{in } \Omega), \qquad (2)$$
$$\phi = \phi_c \qquad (\text{on } \Gamma), \qquad (5)$$
$$\phi = \phi_\infty \qquad (\text{at } \Gamma_\infty). \qquad (6)$$

3.3 Necessary Condition

In this subsection, we derive a necessary condition of optimality, which, in turn, is the sensitivity analysis.

Let $\vec{\tau}(s) = (\tau_z, \tau_x)$ be the tangential vector directed counterclockwise at s. Then, $\tau_z = -n_x, \tau_x = n_z$. We can easily see that

$$\frac{d\vec{n}}{ds} = \frac{1}{R}\vec{\tau}; \qquad (7)$$

hence, in turn

$$\frac{d\vec{\tau}}{ds} = \left(-\frac{dn_x}{ds}, \frac{dn_z}{ds}\right) = -\frac{1}{R}\vec{n}, \qquad (8)$$

where R denotes the radius of curvature at s. On Γ the formula of integration by parts reads

$$\oint_\Gamma f'(s)g(s)ds = \oint_\Gamma f(s)g'(s)ds. \qquad (9)$$

Let us introduce a variation of Γ. Let $\rho(s)$ be a smooth function of s. Let ϵ be a number; its absolute value is small enough. We place segment $\epsilon\rho(s)$ on the normal $\vec{n}$ at s such that positive $\epsilon\rho(s)$ lies on the normal $\vec{n}$. If $|\epsilon|$ is small enough, the end points of the segments will form a closed curve Γ_ϵ which and Γ_∞ enclose a new domain Ω_ϵ. When we consider the following boundary-value problem:

$$\Delta\phi^\epsilon = \alpha\frac{\partial\phi^\epsilon}{\partial z} \qquad ((z,x)\in\Omega_\epsilon); \qquad (10)$$
$$\phi^\epsilon = \phi_c(\text{const.}) \qquad ((z,x)\in\Gamma_\epsilon); \qquad (11)$$
$$\phi^\epsilon = \phi_\infty \qquad ((z,x)\in\Gamma_\infty), \qquad (12)$$

we can easily find that the first variation Θ of ϕ defined by

$$\phi^\epsilon - \phi = \epsilon\Theta + o(\epsilon) \qquad (13)$$

is the solution of

$$\Delta\Theta = \alpha\frac{\partial\Theta}{\partial z} \quad (\text{in } \Omega), \qquad (14)$$
$$\Theta = -\frac{\partial\phi}{\partial n}\rho \quad (\text{on } \Gamma), \qquad (15)$$
$$\Theta = 0 \quad (\text{at } \Gamma_\infty). \qquad (16)$$

On the other hand, we see that the corresponding $\vec{n}_\epsilon$ is given by

$$\vec{n^\epsilon} = \vec{n} + (-\epsilon\rho'(s)\vec{\tau}) + o(\epsilon) \qquad (17)$$

through geometrical inspection. Similarly, we obtain

$$ds^\epsilon = \left(1 + \frac{\epsilon\rho}{R} + o(\epsilon)\right)ds. \qquad (18)$$

Objective functional J^ϵ for ϕ^ϵ is given by

$$J^\epsilon = -\oint_{\Gamma_\epsilon} \mathrm{grad}\phi^\epsilon\cdot\vec{n^\epsilon}ds^\epsilon + \alpha\oint_{\Gamma_\epsilon}\phi^\epsilon n_z^\epsilon ds^\epsilon. \qquad (19)$$

The first term on the right side of (19) is transformed as follows.

$$\begin{aligned}
&\oint_{\Gamma_\epsilon} \mathrm{grad}\phi^\epsilon\cdot ds^\epsilon \\
&= \oint_{\Gamma_\epsilon} \mathrm{grad}(\phi + \epsilon\Theta + o(\epsilon))\cdot\vec{n^\epsilon}ds^\epsilon \\
&= \oint_\Gamma \mathrm{grad}\phi\cdot\vec{n}ds + \epsilon\oint_\Gamma\Big\{\frac{1}{R}\mathrm{grad}\phi\cdot\vec{n} \\
&\quad + \left(\frac{\partial^2\phi}{\partial z^2}n_z^2 + 2\frac{\partial^2\phi}{\partial z\partial x}n_z n_x + \frac{\partial^2\phi}{\partial x^2}n_x^2\right)\Big\}ds \\
&\quad + \epsilon\oint_{\Gamma_\epsilon}\mathrm{grad}\Theta\cdot\vec{n}ds \\
&\quad - \epsilon\oint_{\Gamma_\epsilon}\rho'(s)\mathrm{grad}\phi\cdot\vec{\tau}ds + o(\epsilon),
\end{aligned} \qquad (20)$$

where we used (17) and (18). In order to rewrite (20), let us introduce an adjoint variable p_a as the solution of the following boundary value problem:

$$\Delta p_a + \alpha\frac{\partial p_a}{\partial z} = 0 \qquad (\text{in } \Omega), \qquad (21)$$
$$p_a = 1 \qquad (\text{on } \Gamma), \qquad (22)$$
$$p_a = 0 \qquad (\text{at } \Gamma_\infty). \qquad (23)$$

Thanks to Green's formula, we can calculate as follows:

$$\int_\Omega(\Theta\Delta p_a - p_a\Delta\Theta)da = \oint_{\Gamma+\Gamma_\epsilon}\left(p_a\frac{\partial\Theta}{\partial n} - \Theta\frac{\partial p_a}{\partial n}\right)ds$$
$$= \oint_\Gamma \mathrm{grad}\Theta\cdot\vec{n}ds + \oint_\Gamma\frac{\partial\phi}{\partial n}\frac{\partial p_a}{\partial n}\rho ds, \qquad (24)$$

where we used (15), (16), (22) and (23). Hence, using

(14), (21), (15) and (16), we obtain

$$\oint_\Gamma \text{grad}\Theta \cdot \overrightarrow{n} ds + \oint_\Gamma \frac{\partial \phi}{\partial n}\frac{\partial p_a}{\partial n}\rho ds$$
$$= \int_\Omega (\Theta\Delta p_a - p_a\Delta\Theta)da$$
$$= -\int_\Omega \left(\alpha\frac{\partial p_a}{\partial z}\Theta + \alpha p_a\frac{\partial \Theta}{\partial z}\right) da$$
$$= -\int_\Omega \text{div}(\overrightarrow{\alpha} p_a\Theta)da$$
$$= \oint_\Gamma p_a\Theta\overrightarrow{\alpha}\cdot\overrightarrow{n} ds + \oint_{\Gamma_\infty} p_a\Theta\overrightarrow{\alpha}\cdot\overrightarrow{n} ds$$
$$= -\oint_\Gamma \alpha\frac{\partial\phi}{\partial n}\rho n_z ds, \tag{25}$$

where $\overrightarrow{\alpha} \equiv (\alpha, 0)$.

On the other hand, by integrating by parts, we obtain

$$-\oint_\Gamma \rho'(s)\text{grad}\phi\cdot\overrightarrow{\tau} ds = \oint_\Gamma \rho(s)\frac{\partial}{\partial s}(\text{grad}\phi\overrightarrow{\tau})ds$$
$$= \oint_\Gamma \left\{\left(\frac{\partial}{\partial s}\text{grad}\phi\right)\cdot\overrightarrow{\tau} - \frac{1}{R}\text{grad}\phi\cdot\overrightarrow{n}\right\}\rho ds. \tag{26}$$

Substituting (25) and (26) into (20), we obtain

$$\oint_{\Gamma_\epsilon} \text{grad}\phi^\epsilon\cdot ds^\epsilon = \oint_\Gamma \text{grad}\phi\cdot\overrightarrow{n} ds$$
$$+\epsilon\oint_\Gamma\left(\frac{\partial^2\phi}{\partial z^2}n_z^2 + 2\frac{\partial^2\phi}{\partial z\partial x}n_z n_x + \frac{\partial^2\phi}{\partial x^2}n_x^2\right)\rho ds$$
$$-\epsilon\oint_\Gamma\left(\frac{\partial\phi}{\partial n}\frac{\partial p_a}{\partial n}\rho + \alpha\frac{\partial\phi}{\partial n}n_z\rho\right)ds$$
$$+\epsilon\oint_\Gamma\left(\frac{\partial}{\partial s}\text{grad}\phi\right)\cdot\overrightarrow{\tau}\rho + o(\epsilon). \tag{27}$$

Similarly, we can see that

$$\oint_{\Gamma_\epsilon}\phi^\epsilon n_z^\epsilon ds^\epsilon = \oint_\Gamma \phi n_z ds$$
$$+\epsilon\oint_\Gamma\left(\frac{1}{R}\phi\rho n_z + \rho\frac{\partial}{\partial s}(\phi\tau_z)\right)ds$$
$$= \oint_\Gamma \phi n_z ds + o(\epsilon), \tag{28}$$

since $(\partial\phi/\partial s)$ vanishes on Γ. From (27) we see that

$$\oint_{\Gamma_\epsilon}\text{grad}\phi^\epsilon\cdot ds^\epsilon - \oint_\Gamma\text{grad}\phi\cdot\overrightarrow{n} ds$$
$$= -\epsilon\oint_\Gamma\Big(\frac{\partial\phi}{\partial n}\frac{\partial p_a}{\partial n}\rho + \alpha\frac{\partial\phi}{\partial n}n_z\rho\Big)ds$$
$$+\epsilon\oint_\Gamma\Big(\frac{\partial^2\phi}{\partial z^2}n_z^2 + 2\frac{\partial^2\phi}{\partial z\partial x}n_z n_x + \frac{\partial^2\phi}{\partial x^2}n_x^2\Big)\rho ds$$
$$+\epsilon\oint_\Gamma\Big(\frac{\partial^2\phi}{\partial z^2}n_z^2 + -2\frac{\partial^2\phi}{\partial z\partial x}n_z n_x + \frac{\partial^2\phi}{\partial x^2}n_x^2\Big)\rho ds$$
$$+o(\epsilon)$$
$$= -\epsilon\oint_\Gamma\Big(\frac{\partial\phi}{\partial n}\frac{\partial p_a}{\partial n}\rho + \alpha\frac{\partial\phi}{\partial n}n_z\rho\Big)ds + \epsilon\alpha\oint_\Gamma\frac{\partial\phi}{\partial z}\rho ds$$
$$+o(\epsilon). \tag{29}$$

Hence, if we define δJ by

$$J^\epsilon - J = \epsilon\delta J + o(\epsilon), \tag{30}$$

we observe that

$$\delta J = -\oint_\Gamma\left(\frac{\partial\phi}{\partial n}\frac{\partial p_a}{\partial n} + \alpha\frac{\partial\phi}{\partial n}n_z - \alpha\frac{\partial\phi}{\partial z}\right)\rho ds. \tag{31}$$

Since admissible cavities must satisfy

$$\int_{\Omega^c} da = S, \tag{32}$$

where S is the given area of cross-section of the cavities, we have

$$\oint_\Gamma \rho ds = 0. \tag{33}$$

If Ω attains minimum q_u, δJ must vanish for every infinitesimally small (say, $\|\rho\| \ll 1$, $\|\cdot\|$ is an appropriate norm) ρ which satisfies (33). However, (33) and (31) is linear in ρ; this means, in turn, δJ must vanish for every ρ that satisfies (33). Thus, we obtain the following necessary condition of minimum, a main result of this paper:

Theorem. *If the cavity boundary Γ attains a minimum q_u, then there exists a constant λ (Lagrange multiplier) such that*

$$\frac{\partial\phi}{\partial n}\frac{\partial p_a}{\partial n} + \alpha n_z\frac{\partial\phi}{\partial n} - \alpha\frac{\partial\theta}{\partial z} = \lambda \tag{34}$$

holds at every point on Γ. Here, ϕ is the solution of (2), (5), (6) *and p_a is the solution of* (21) - (23).

4. CONCLUSION

The solution of (2), (5), (6) and the solution of (21) – (23) are given by infinite series expansions in terms of modified Bessel functions of second type; it is difficult to test the condition (34) for these solution analytically. However, (31) gives the gradient of the objective function; (31) can be used for numerical calculation of the optimal shape. Numerical methods for shape optimization have not been well developed. The study of the methods is left yet. As for more general and prototype shape optimization, the readers can refer Fujii (1990). As for the sufficient conditions, the readers can refer Belov and Fujii (1997).

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